Nonreflecting Boundary Conditions for Euler Equation Calculations

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This paper presents a unified theory for the construction of steady-state and unsteady nonreflecting boundary conditions for the Euler equations. These allow calculations to be performed on truncated domains without the generation of spurious nonphysical reflections at the far-field boundaries. The general theory, developed previously by mathematicians, is presented in a more easily understood form based upon fundamental ideas of linear analysis. The application to the Euler equations is given, and the relation to standard "quasi-one-dimensional" boundary conditions is explained. Results for turbomachinery problems show the effectiveness of the new boundary conditions, particularly the steady-state nonreflecting boundary conditions.

I. Introduction

THE objective in formulating nonreflecting boundary conditions is to prevent spurious, nonphysical reflections at inflow and outflow boundaries, so that the calculated flowfield is independent of the location of the far-field boundaries. This leads to greater accuracy and greater computational efficiency, since the computational domain can be made much smaller.

The theoretical basis for nonreflecting boundary conditions stems from a paper by Engquist and Majda,[1] which discusses both ideal nonreflecting boundary conditions and a method for constructing approximate forms, and a paper by Kreiss,[2] which analyzes the wellposedness of initial boundary value problems for hyperbolic systems. Many workers have been active in this area in the last ten years, but their work has been mainly concerned with scalar partial differential equations, with only a couple of recent applications to the Euler equations in specific circumstances.[3,4] Also, almost all of the literature has been written by mathematicians, and in their desire to be absolutely rigorous in their analysis, they use a formalism and assume a background foundation in advanced differential equation theory that makes it difficult for the papers to be appreciated by those with an engineering background.

The author has recently completed a lengthy report on the formulation of nonreflecting boundary conditions and the application to the Euler equations.[5] This report presents a unified view of the theory, with some extensions required for the Euler equations, and does so using the simplest approach possible based upon linear analysis. In taking this approach some rigor is sacrificed, and the conditions for wellposedness become necessary, but possibly not sufficient. The report also shows in full detail the application of the theory to the Euler equations. Another report describes the details of the implementation of the numerical boundary conditions[6] for two-dimensional turbomachinery applications.

The purpose of this paper is to summarize the principal parts of these two reports, and to present results that demonstrate the effectiveness of the new boundary conditions in turbomachinery applications. Because of space limitations, all of the wellposedness analysis, a large amount of algebraic detail, some interesting additional applications, and a variety of helpful comments and insights have been omitted from this paper; the interested reader is urged to refer to the original two reports[5,6] to obtain these.

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II. General Analysis

A. Fourier Analysis

In two dimensions, the analysis is concerned with the following time-dependent, hyperbolic partial differential equation:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \tag{1}$$

where U is an N-component column vector and A and B are constant $N \times N$ matrices. We consider wave-like solutions of the form

$$U(x,y,t) = e^{i(kx + ly - \omega t)}\boldsymbol{u}^R \tag{2}$$

where $\boldsymbol{u}^R$ is a constant column vector. Substituting this into the differential equation (1), we find that

$$(-\omega I + kA + lB)\boldsymbol{u}^R = 0 \tag{3}$$

which has nontrivial solutions, provided that

$$\det(-\omega I + kA + lB) = 0 \tag{4}$$

Equation (4) is called the dispersion relation, and it is a polynomial equation of degree N in each of ω, k, and l. We will be concerned with the roots k_n of this equation for given values of ω and l. By dividing the dispersion relation by ω, we obtain

$$\det\left(-I + \frac{k_n}{\omega}A + \frac{l}{\omega}B\right) = 0 \tag{5}$$

and so it is clear that k_n/ω is a function of l/ω. Thus, the variable $\lambda = l/\omega$ will play a key role in constructing all of the boundary conditions.

A critical step in the construction and analysis of boundary conditions is to separate the waves into incoming and outgoing modes. If ω is complex with $\mathrm{Im}(\omega)>0$ (giving an exponential growth in time), then the right-propagating waves are those for which $\mathrm{Im}(k)>0$. This is because the amplitude of each wave is proportional to $e^{\mathrm{Im}(\omega)(t-x/c)}$, where $c = \mathrm{Im}(\omega)/\mathrm{Im}(k)$ is the apparent velocity of propagation.

If ω and k are real, then a standard result in the analysis of dispersive wave propagation[7] is that the velocity of energy propagation is the group velocity defined by

$$c_g = \begin{pmatrix} \partial\omega/\partial k \\ \partial\omega/\partial l \end{pmatrix} \tag{6}$$

Hence, for real ω the incoming waves are those that either have $\mathrm{Im}(k)>0$, or have $\mathrm{Im}(k)=0$ and $\partial\omega/\partial k>0$.

The column vector u^R is the right null-vector of the singular matrix $(-\omega I + kA + lB)$. The construction of the nonreflecting boundary conditions requires the row vector v^L, which is the left null-vector of the singular matrix $A^{-1}(-\omega I + kA + lB)$.

$$v^L A^{-1}(-\omega I + kA + lB) = 0 \tag{7}$$

One of the important features of this left null-vector is its orthogonality to u^R. If k_m and k_n are two different solutions to the dispersion relation for the same values of ω and l, and if u_m^R and v_n^L are the corresponding right and left eigenvectors, then

$$v_n^L A^{-1}(-\omega I + k_m A + lB)u_m^R = 0 \tag{8}$$

and

$$v_n^L A^{-1}(-\omega I + k_n A + lB)u_m^R = 0 \tag{9}$$

Subtracting one from the other gives

$$(k_m - k_n)v_n^L u_m^R = 0 \Rightarrow v_n^L u_m^R = 0 \tag{10}$$

B. Ideal Nonreflecting Boundary Conditions

Suppose that the differential equation is to be solved in the domain $x>0$, and one wants to construct nonreflecting boundary conditions at $x=0$ to minimize or ideally prevent the reflection of outgoing waves. At the boundary $x=0$, U can be decomposed into a sum of Fourier modes with different values of ω and l, so the analysis begins by considering just one particular choice of ω and l. In this case, the most general form for U is

$$U(x,y,t) = \left[\sum_{n=1}^{N} a_n u_n^R e^{ik_n x}\right] e^{i(ly-\omega t)} \tag{11}$$

where k_n is the nth root of the dispersion relation for the given values of ω and l, and u_n^R is the corresponding right eigenvector.

The ideal nonreflecting boundary conditions would be to specify that $a_n = 0$ for each n that corresponds to an incoming wave. Because of orthogonality,

$$v_n^L U = v_n^L \left[\sum_{m=1}^{N} a_m u_m^R e^{ik_m x}\right] e^{i(ly-\omega t)}$$

$$= a_n \left(v_n^L u_n^R\right) e^{ik_n x} e^{i(ly-\omega t)} \tag{12}$$

and so an equivalent specification of nonreflecting boundary conditions is

$$v_n^L U = 0 \tag{13}$$

for each n corresponding to an incoming mode.

In principle, these exact boundary conditions can be implemented in a numerical method. The problem is that, in general, v_n^L depends on λ and so the implementation would involve a Fourier transform in y and a Laplace transform in t. Computationally, this is both difficult and expensive to implement. In situations in which there is only one known frequency, it is possible to use the ideal boundary conditions, and this has been done for linearized, unsteady potential and Euler equations.[8-10] The remainder of this paper is concerned with three types of approximation that can be used in the more general situation, without requiring a Laplace transformation.

C. One-Dimensional, Unsteady Boundary Conditions

The one-dimensional nonreflecting boundary condition is obtained by ignoring all variations in the y direction and setting $\lambda = l/\omega = 0$. The corresponding right and left eigenvectors are important in defining and implementing the other boundary conditions, and so we label them w, with

$$w_n^R = u_n^R|_{\lambda=0} \tag{14}$$

and

$$w_n^L = v_n^L|_{\lambda=0} \tag{15}$$

The one-dimensional boundary condition, expressed in terms of the primitive variables, is

$$w_n^L U = 0 \tag{16}$$

for all n corresponding to incoming waves. If the right and left eigenvectors are normalized so that

$$w_m^L w_n^R = \delta_{mn} \equiv \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{17}$$

then they can be used to define a transformation between the primitive variables and the one-dimensional characteristic variables, i.e.,

$$U = \sum_{n=1}^{N} c_n w_n^R \tag{18}$$

where

$$c_n = w_n^L U \tag{19}$$

Expressed in terms of the characteristic variables, the boundary condition is simply that $c_n = 0$ for each incoming wave.

D. Exact, Two-Dimensional, Steady Boundary Condition

The exact, two-dimensional, steady boundary condition may be considered to be the limit of the ideal boundary condition, as $\omega \to 0$. Performing a Fourier decomposition of the solution (which is assumed to be periodic in y with period 2π), we find that

$$U(0,y,t) = \sum_{l=\infty}^{\infty} \hat{U}_l(t) e^{ily} \tag{20}$$

where

$$\hat{U}_l(t) = \frac{1}{2\pi}\int_0^{2\pi} U(0,y,t) e^{-ily}\, dy \tag{21}$$

The boundary condition for $l \neq 0$ is

$$s_n^L \hat{U}_l = 0 \tag{22}$$

for each incoming wave n, where

$$s_n^L = \lim_{\lambda\to\infty} v_n^L(\lambda) \tag{23}$$

The boundary condition for the $l=0$ mode, which is the solution average at the boundary, is

$$v_n^L(0)\hat{U}_0 = 0 \Rightarrow w_n^L \hat{U}_0 = 0 \tag{24}$$

for each incoming wave n. The right side of Eq. (24) can be modified by the user to specify the value of the incoming average characteristics. Further discussion of this point will be delayed until the Sec. III.

E. Approximate, Two-Dimensional, Unsteady Boundary Condition

A sequence of approximate, nonreflecting boundary conditions can be obtained by expanding v_n^L in a Taylor series as a function of l/ω^1, i.e.,

$$v_n^L(\lambda) = v_n^L|_{\lambda=0} + \lambda \frac{dv_n^L}{d\lambda}\bigg|_{\lambda=0} + \frac{1}{2}\lambda^2 \frac{d^2 v_n^L}{d\lambda^2}\bigg|_{\lambda=0} + \ldots \tag{25}$$

The first-order approximation, obtained by keeping only the leading term, gives the one-dimensional boundary condition. The second-order approximation is

$$\bar{v}_n^L(\lambda) = v_n^L|_{\lambda=0} + \frac{l}{\omega}\frac{dv_n^L}{d\lambda}\bigg|_{\lambda=0} \tag{26}$$

where the overbar denotes that $\bar{v}$ is an approximation to v. This produces the boundary condition

$$\left(v_n^L|_{\lambda=0} + \frac{l}{\omega}\frac{dv_n^L}{d\lambda}\bigg|_{\lambda=0} \right) U = 0 \tag{27}$$

Multiplying by $-i\omega$ and replacing $i\omega$ and il by $-\partial/\partial t$ and $\partial/\partial y$, respectively, gives

$$v_n^L|_{\lambda=0} \frac{\partial U}{\partial t} - \frac{dv_n^L}{d\lambda}\bigg|_{\lambda=0} \frac{\partial U}{\partial y} = 0 \tag{28}$$

This is a local boundary condition of the same differential order as the governing equations, and so it can, in general, be implemented without difficulty. It must be emphasized that these boundary conditions are only approximately nonreflecting and may produce significant nonphysical reflections of outgoing waves for which λ is far from zero. In particular, it can be shown that for a given wavenumber l, the boundary conditions are perfectly reflective at the critical cut-off frequency at which the normal group velocity component is zero.[5]

F. Wellposedness and Reflection Analysis

Wellposedness is the requirement that a solution exists, is unique, and is bounded in the sense that small perturbations in the boundary data produce small changes in the solution. Any hyperbolic system arising from a model of a physical problem ought to be wellposed, and so it is critical that the far-field boundary conditions used to truncate the solution domain give a wellposed problem. Higdon has written an excellent review[11] of the work of Kreiss[2] and others and, in particular, gives a physical interpretation of the theory in terms of wave propagation.

Because of space limitations, this theory is not presented here. The basic idea behind it is that if there is an incoming wave that exactly satisfies the boundary conditions, then it can grow without bound and so the problem is ill posed. Using an energy argument, it can be shown that if A and B are symmetric, or can be simultaneously symmetrized, the one-dimensional boundary conditions are always well posed. However, for the higher order nonreflecting boundary conditions, no such general result exists, and each application must be analyzed separately. In the case of the Euler equations, there are difficulties in the analysis because of two different types of degeneracy. Additional theory to overcome these problems is presented in one of the two original reports.[5]

A slight variation on the wellposedness analysis assesses the effectiveness of the boundary conditions, by considering a general solution that is a sum of incoming and outgoing modes. The amplitudes of the incoming modes can be expressed as functions of the amplitudes of the outgoing modes, using reflection coefficients. In the ideal case, these coefficients are zero. Using one-dimensional boundary conditions, the coefficients are $O(l/\omega)$, and using the approximate, unsteady boundary conditions they are in general $O(l/\omega)^2$.

III. Application to Euler Equations

A. Fourier Analysis

The linearized, two-dimensional Euler equations can be written in terms of primitive variables as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \tag{29}$$

where

$$U = \begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} \tag{30}$$

$$A = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/P \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/P \\ 0 & 0 & \gamma p & v \end{pmatrix} \tag{31}$$

The elements of the vector U represent perturbations from uniform flow conditions, and the matrices A and B are evaluated using these same conditions. The analysis is greatly simplified if the unsteady perturbations and the steady variables in A and B are all nondimensionalized, using the steady density and speed of sound. With this choice of nondimensionalization, the final forms of the matrices A and B are

$$A = \begin{pmatrix} u & 1 & 0 & 0 \\ 0 & u & 0 & 1 \\ 0 & 0 & u & 0 \\ 0 & 1 & 0 & u \end{pmatrix}, \quad B = \begin{pmatrix} v & 0 & 1 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1 \\ 0 & 0 & 1 & v \end{pmatrix} \tag{32}$$

and the variables u and v in the preceding matrices are now the Mach numbers in the x and y directions.

Following the analytic theory described earlier, we first obtain the dispersion relation

$$(uk + vl - \omega)^2[(uk + vl - \omega)^2 - k^2 - l^2] = 0 \tag{33}$$

Two of the four roots are clearly identical, i.e.,

$$k_{1,2} = \frac{\omega - vl}{u} \tag{34}$$

For $u > 0$, these correspond to right-traveling waves.

The other two roots are determined from the equation

$$(1 - u^2)k^2 - 2u(vl - \omega)k - (vl - \omega)^2 + l^2 = 0 \tag{35}$$

Hence, the third and fourth roots are defined by

$$k_3 = \frac{(\omega - vl)(-u + S)}{1 - u^2} \tag{36}$$

$$k_4 = \frac{(\omega - vl)(-u - S)}{1 - u^2} \tag{37}$$

where

$$S = \sqrt{1 - (1 - u^2) l^2 / (\omega - vl)^2} \tag{38}$$

For $0 < u < 1$, which corresponds to subsonic flow normal to the boundary, the third root is a right-traveling wave, and the fourth root is a left-traveling wave, provided the correct branch of the complex square root function is used in defining S. It can be shown[5] that if ω is real and S^2 is real and positive, then the positive root must be taken, whereas if ω and/or S are complex then the complex roots must be chosen such that k_3 has a positive imaginary component and k_4 has a negative imaginary component.

B. Eigenvectors

1. *Root 1: Entropy Wave*

After some algebra,[5] it can be shown that the appropriate right and left eigenvectors for the root

$$k_1 = \frac{\omega - vl}{u} \tag{39}$$

are

$$u_1^R = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{40}$$

and

$$v_1^L = (-1 \ \ 0 \ \ 0 \ \ 1) \tag{41}$$

This choice of eigenvectors corresponds to an entropy wave. This can be verified by noting that the only nonzero term in the right eigenvector is the density, so that the wave has varying entropy, no vorticity, and constant pressure. Also, the left eigenvector "measures" entropy in the sense that $v_1^L U$ is equal to the linearized entropy $\delta p - \delta \rho$ (remembering that $c = 1$ because of the nondimensionalization).

2. *Root 2: Vorticity Wave*

The second set of right and left eigenvectors for the multiple root

$$k_2 = \frac{\omega - vl}{u} \tag{42}$$

are

$$u_2^R = \begin{pmatrix} 0 \\ -ul/\omega \\ uk_2/\omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -u\lambda \\ 1 - v\lambda \\ 0 \end{pmatrix} \tag{43}$$

and

$$v_2^L = (0 \ \ -u\lambda \ \ 1 - v\lambda \ \ -\lambda) \tag{44}$$

This root corresponds to a vorticity wave, which can be verified by noting that the right eigenvector gives a wave with vorticity, but uniform entropy and pressure.

3. *Root 3: Downstream Running Wave*

The eigenvectors for

$$k_3 = \frac{(\omega - vl)(S - u)}{1 - u^2} \tag{45}$$

are

$$u_3^R = \frac{1+u}{2\omega} \begin{pmatrix} \omega - uk_3 - vl \\ k_3 \\ l \\ \omega - uk_3 - vl \end{pmatrix} = \frac{1}{2(1-u)} \begin{pmatrix} (1 - r\lambda)(1 - uS) \\ (1 - v\lambda)(S - u) \\ (1 - u^2)\lambda \\ (1 - r\lambda)(1 - uS) \end{pmatrix} \tag{46}$$

and

$$v_3^L = [0 \ \ (1 - v\lambda) \ \ u\lambda \ \ (1 - v\lambda)S] \tag{47}$$

This root corresponds to an isentropic, irrotational pressure wave traveling downstream.

4. *Root 4: Upstream Running Pressure Wave*

The eigenvectors for

$$k_4 = \frac{(\omega - vl)(S + u)}{1 - u^2} \tag{48}$$

are

$$u_3^R = \frac{1-u}{2\omega} \begin{pmatrix} \omega - uk^4 - vl \\ k_4 \\ l \\ \omega - uk_4 - vl \end{pmatrix} = \frac{1}{2(1+u)} \begin{pmatrix} (1 - v\lambda)(1 + uS) \\ -(1 - v\lambda)(S + u) \\ (1 - u^2)\lambda \\ (1 - v\lambda)(1 + uS) \end{pmatrix} \tag{49}$$

and

$$v_4^L = [0 \ \ -(1 - v\lambda) \ \ -u\lambda \ \ (1 - v\lambda)S] \tag{50}$$

This root corresponds to an isentropic, irrotational pressure wave traveling upstream provided $u < 1$.

C. One-Dimensional, Unsteady Boundary Conditions

If the computational domain is $0 < x < 1$ and $0 < u < 1$, then the boundary at $x = 0$ is an inflow boundary with incoming waves corresponding to the first three roots, and the boundary at $x = 1$ is an outflow boundary with just one incoming wave due to the fourth root.

When $\lambda = 0$, $S = 1$, and so the right eigenvectors w^R are

$$w_1^R = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2^R = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$w_3^R = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \quad w_4^R = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \end{pmatrix} \tag{51}$$

and the left eigenvectors w^L are

$$\begin{aligned} w_1^L &= (-1 \quad 0 \quad 0 \quad 1) \\ w_2^L &= (\ \ 0 \quad 0 \quad 1 \quad 0) \\ w_3^L &= (\ \ 0 \quad 1 \quad 0 \quad 1) \\ w_4^L &= (\ \ 0 \ \ -1 \quad 0 \quad 1) \end{aligned} \tag{52}$$

Hence the transformation to, and from, one-dimensional characteristic variables is given by the following two matrix equations.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} \tag{53}$$

$$\begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} = \begin{pmatrix} -1 & 0 & ½ & ½ \\ 0 & 0 & ½ & -½ \\ 0 & 1 & 0 & 0 \\ 0 & 0 & ½ & ½ \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{54}$$

where $\delta\rho$, δu, δv, and δp are the perturbations from the uniform flow about which the Euler equations were linearized, and c_1, c_2, c_3, and c_4 are the amplitudes of the four characteristic waves. At the inflow boundary, the correct unsteady, nonreflecting boundary conditions are

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \tag{55}$$

whereas at the outflow boundary the correct nonreflecting boundary condition is

$$c_4 = 0 \tag{56}$$

The standard numerical method for implementing these is to calculate or extrapolate the outgoing characteristic values from the interior domain, and then use Eq. (54) to reconstruct the solution on the boundary.

D. Exact, Two-Dimensional, Steady Boundary Conditions

The exact, two-dimensional, steady boundary conditions are essentially the ideal boundary conditions in the limit $\omega \to 0$. One begins by performing a Fourier decomposition of U along the boundary:

$$U(0,y,t) = \sum_{-\infty}^{\infty} \hat{U}_m(t) e^{il_m y} \tag{57}$$

where

$$\hat{U}_m(t) = \frac{1}{P} \int_0^P U(0,y,t) e^{-il_m y} \, dy \tag{58}$$

and

$$l_m = \frac{2\pi m}{P} \tag{59}$$

Boundary conditions are now constructed for each Fourier mode. If the mode number m is nonzero, then

$$\lim_{\lambda \to \infty} S(\lambda) = \sqrt{1 - \frac{1-u^2}{v^2}} = -\frac{\beta}{v} \tag{60}$$

where

$$\beta = \begin{cases} i \, \mathrm{sign}(l) \sqrt{1-u^2-v^2}, & u^2+v^2<1 \\ -\mathrm{sign}(v) \sqrt{u^2+v^2-1}, & u^2+v^2>1 \end{cases} \tag{61}$$

The reason for the choice of sign functions in the definition of β is that for supersonic flow, S must be positive, as discussed when S was first defined, and for subsonic flow, S must be consistent with $\mathrm{Im}(k_3) > 0$.

The next step is to construct the steady-state left eigenvectors s^L. Because it is permissible to multiply the eigenvectors by any function of λ, we will slightly modify the definition given in the theory section in order to keep the limits finite as $\lambda \to \infty$.

$$\begin{aligned} s_1^L &= \lim_{\lambda \to \infty} v_1^L = (-1 \quad 0 \quad 0 \quad 1) \\ s_2^L &= \lim_{\lambda \to \infty} \tfrac{1}{\lambda} v_2^L = (0 \quad -u \quad -v \quad -1) \\ s_3^L &= \lim_{\lambda \to \infty} \tfrac{1}{\lambda} v_3^L = (0 \quad -v \quad u \quad \beta) \\ s_4^L &= \lim_{\lambda \to \infty} \tfrac{1}{\lambda} v_4^L = (0 \quad v \quad -u \quad \beta) \end{aligned} \tag{62}$$

Using these vectors, the exact two-dimensional, steady-state nonreflecting boundary conditions at the inflow are

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -u & -v & -1 \\ 0 & -v & u & \beta \end{pmatrix} \hat{U}_m = 0 \tag{63}$$

and at the outflow the boundary condition is

$$(0 \;\; v \;\; -u \;\; \beta) \, \hat{U}_m = 0 \tag{64}$$

For subsonic flow, β depends on l and, hence, the mode number m. For supersonic flow, β does not depend on l and so the boundary conditions are the same for each Fourier mode other than $m = 0$.

In order to discuss the approach in implementing these conditions, we now transform from primitive to characteristic variables. The inflow boundary condition becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & -½(1+u) & -½(1-u) \\ 0 & u & ½(\beta - v) & ½(\beta + v) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0 \tag{65}$$

and the outflow equation becomes

$$[0 \;\; -u \;\; ½(\beta + v) \;\; ½(\beta - v)] \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0 \tag{66}$$

After solving to obtain the incoming characteristics as a function of the outgoing ones, we find that

$$\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\left(\dfrac{\beta + v}{1+u}\right)\hat{c}_4 \\ \left(\dfrac{\beta + v}{1+u}\right)\hat{c}_4 \end{pmatrix} \tag{67}$$

and

$$\hat{c}_4 = \left(\frac{2u}{\beta - v}\right)\hat{c}_2 - \left(\frac{\beta + v}{\beta - v}\right)\hat{c}_3 \tag{68}$$

It has already been stated that if the incoming characteristic variables are set to zero, then the initial-boundary-value problem is wellposed. This suggests that the evolutionary process for the steady-state problem will be wellposed if we lag the updating of the incoming characteristics, i.e., we set

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat c_1\\ \hat c_2\\ \hat c_3\end{pmatrix} = \alpha \begin{pmatrix} -\hat c_1 \\ -\left(\dfrac{\beta+v}{1+u}\right)\hat c_4 - \hat c_2 \\ \left(\dfrac{\beta+v}{1+u}\right)^2 \hat c_4 - \hat c_3 \end{pmatrix} \tag{69}$$

$$\frac{\partial \hat c_4}{\partial t} = \alpha\left[\left(\frac{2u}{\beta-v}\right)\hat c_2 - \left(\frac{\beta+v}{\beta-v}\right)\hat c_3 - \hat c_4\right] \tag{70}$$

Numerical experience indicates that a suitable choice for α is $1/P$. This completes the formulation of the boundary conditions for all of the Fourier modes except $m = 0$, which corresponds to the $l = 0$ average mode. For this mode, the user specifies the changes in the incoming one-dimensional characteristics in order to achieve certain average flow conditions.

In the turbomachinery program developed by the author, the three incoming characteristics are determined by specifying the averge entropy, flow angle, and stagnation enthalpy at the inflow boundary, and at the outflow boundary the one incoming characteristic is determined by specifying the average exit pressure. Full details of this numerical procedure are given in Ref. 6, which explains how nonlinearities lead to second-order nonuniformities in entropy and stagnation enthalpy across the inflow boundary. These are undesirable, and are avoided by modifying one of the inflow boundary conditions, and replacing another by the constraint of uniform stagnation enthalpy. The report also shows how the same boundary condition approach can be used to match together two stator and rotor calculations, so that the interface is treated in an average, conservative manner.

E. Approximate, Two-Dimensional, Unsteady Boundary Conditions

1. Second-Order Boundary Conditions

Following the theory presented earlier, the second-order, nonreflecting boundary conditions are obtained by taking the second-order approximation to the left eigenvectors v^L in the limit $\lambda \approx 0$. In this limit, $S \approx 1$, and so one obtains the following approximate eigenvectors.

$$\begin{aligned}
\bar v_1^L &= (-1 \quad 0 \quad 0 \quad 1)\\
\bar v_2^L &= (0 \quad -u\lambda \quad 1-v\lambda \quad -\lambda)\\
\bar v_3^L &= (0 \quad 1-v\lambda \quad u\lambda \quad 1-v\lambda)\\
\bar v_4^L &= [0 \quad -(1-v\lambda) \quad -u\lambda \quad 1-v\lambda]
\end{aligned} \tag{71}$$

Actually, the first two eigenvectors are exact, since the only approximation that has been made is setting $S \approx 1$ in the third fourth eigenvectors. Consequently, the inflow boundary conditions will be perfectly nonreflecting for both of the incoming entropy and vorticity characteristics.

The second step is to multiply by $i\omega$ and replace $i\omega$ by $-\partial/\partial t$ and il by $\partial/\partial y$. This gives the inflow boundary condition

$$\begin{pmatrix} -1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\frac{\partial U}{\partial t} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & u & v & 1\\ 0 & v & -u & v\end{pmatrix}\frac{\partial U}{\partial y} = 0 \tag{72}$$

and the outflow boundary condition

$$(\,0 \quad -1 \quad 0 \quad 1\,)\frac{\partial U}{\partial t} + (\,0 \quad -v \quad u \quad v\,)\frac{\partial U}{\partial y} = 0 \tag{73}$$

For implementation, it is preferable to rewrite these equations using one-dimensional characteristics.

$$\frac{\partial}{\partial t}\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & v & (1+u)/2 & (1-u)/2\\ 0 & -u & v & 0\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = 0 \tag{74}$$

$$\frac{\partial c_4}{\partial t} + (\,0 \quad u \quad 0 \quad v\,)\frac{\partial}{\partial y}\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = 0 \tag{75}$$

Before actually implementing these boundary conditions, a wellposedness analysis was performed. This revealed that the outflow boundary condition is well posed, but the inflow boundary condition is ill posed, with an incoming wave that grows exponentially. Hence, it was necessary to modify the inflow boundary condition.

2. Modified Boundary Conditions

To overcome the illposedness of the inflow boundary conditions, we modified the third inflow boundary condition. To do this, we noted that we are overly restrictive in requiring v_3^L to be orthogonal to u_1^R and u_2^R. Because the first two inflow boundary conditions already require that $a_1 = a_2 = 0$, we only really require that v_3^L is orthogonal to u_4^R. Thus, we proposed a new definition of $\bar v_3^L$ that is equal to $(\bar v_3^L)_{\rm old}$ plus λ times some multiple of the leading order term in $\bar v_2^L$.

$$\bar v_3^L = (0 \quad 1 \quad u\lambda \quad 1) + \lambda m(0 \quad 0 \quad 1 \quad 0) \tag{76}$$

The variable m was chosen to minimize $\bar v_3^L u_4^R$, which controls the magnitude of the reflection coefficient. The motivation for this approach was that the second approximation to the scalar wave equation is wellposed and produces fourth-order reflections.[1] Carrying out this procedure resulted in the following modified inflow boundary condition.

$$\begin{pmatrix} -1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\frac{\partial U}{\partial t} + \begin{pmatrix} -v & 0 & 0 & v\\ 0 & u & v & 1\\ 0 & v & (1-u)/2 & v\end{pmatrix}\frac{\partial U}{\partial y} = 0 \tag{77}$$

Analysis confirmed that this is well posed.

Finally, it is helpful to express this boundary condition in characteristic form, i.e.,

$$\frac{\partial}{\partial t}\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix} + \begin{pmatrix} v & 0 & 0 & 0\\ 0 & v & (1+u)/2 & (1-u)/2\\ 0 & (1-u)/2 & v & 0\end{pmatrix}\frac{\partial}{\partial y} \times \begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = 0 \tag{78}$$

The numerical implementation of this boundary condition is straightforward. In the program developed by the author, the changes in the outgoing characteristics are obtained from the changes distributed by Ni's version of the Lax-Wendroff algorithm, which is used to solve the unsteady Euler equations on the interior domain. The changes in the incoming characteristics are calculated by integrating the boundary conditions, using a one-dimensional Lax-Wendroff algorithm. The combined characteristic changes are then converted back into changes in the primitive variables, and, hence, the conservation variables.[6] The outflow boundary condition is implemented in a similar fashion.

F. Dimensional Boundary Conditions

For convenience, this section lists all of the boundary conditions in the original dimensional variables.

1) Transformation to, and from, one-dimensional characteristic variables.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -c^2 & 0 & 0 & 1 \\ 0 & 0 & \rho c & 0 \\ 0 & \rho c & 0 & 1 \\ 0 & -\rho c & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} \tag{79}$$

$$\begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} = \begin{pmatrix} -1/(c^2) & 0 & 1/(2c^2) & 1/(2c^2) \\ 0 & 0 & 1/(2\rho c) & -1/(2\rho c) \\ 0 & 1/(\rho c) & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{80}$$

2) One-dimensional, unsteady boundary conditions.
Inflow:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \tag{81}$$

Outflow:

$$c_4 = 0 \tag{82}$$

3) Exact, two-dimensional, steady boundry conditions.
Inflow:

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \alpha \begin{pmatrix} -\hat{c}_1 \\ -\left(\dfrac{c\beta + v}{c + u}\right)\hat{c}_4 - \hat{c}_2 \\ \left(\dfrac{c\beta + v}{c + u}\right)^2 \hat{c}_4 - \hat{c}_3 \end{pmatrix} \tag{83}$$

$$\frac{\partial \hat{c}_4}{\partial t} = \alpha\left[\left(\frac{2u}{c\beta - v}\right)\hat{c}_2 - \left(\frac{c\beta + v}{c\beta - v}\right)\hat{c}_3 - \hat{c}_4\right] \tag{84}$$

where

$$\beta = \begin{cases} i \operatorname{sign}(l)\sqrt{1 - M^2}, & M < 1 \\ -\operatorname{sign}(v)\sqrt{M^2 - 1}, & M > 1 \end{cases} \tag{85}$$

4) Fourth-order, two-dimensional, unsteady, inflow boundary condition.

$$\frac{\partial}{\partial t}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & (c+u)/2 & (c-u)/2 \\ 0 & (c-u)/2 & v & 0 \end{pmatrix} \times \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{86}$$

5) Second-order, two-dimensional, unsteady, outflow boundary condition.

$$\frac{\partial c_4}{\partial t} + (0 \;\; u \;\; 0 \;\; v)\frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{87}$$

IV. Results

A. Steady, Nonreflecting Boundary Conditions

To verify the effectiveness of the steady-state, nonreflecting boundary conditions, Figs. 1 and 2 show results for a high-turning turbine cascade. Figure 1 shows results for subsonic outflow conditions for two different locations of the far-field

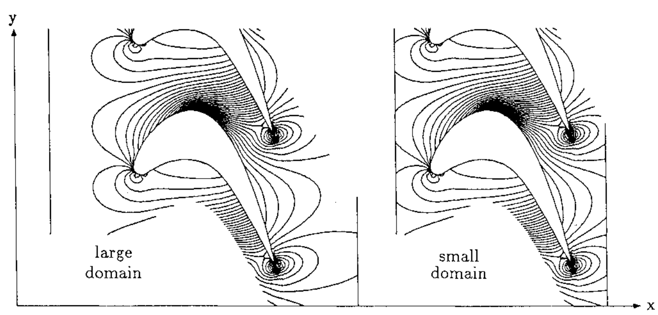

Fig. 1 Pressure contours using nonreflecting boundary conditions, $M_{exit} = 0.75$.

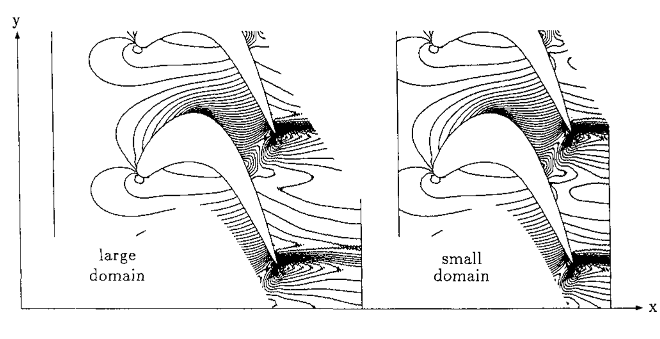

Fig. 2 Pressure contours using nonreflecting boundary conditions, $M_{exit} = 1.1$.

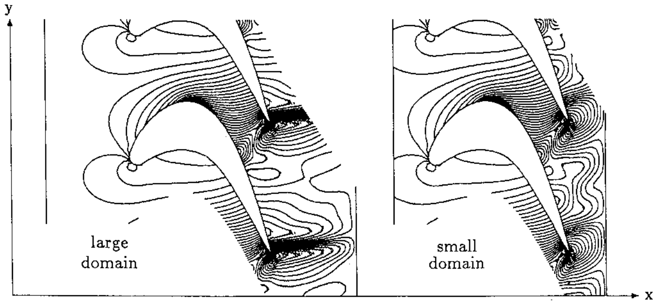

Fig. 3 Pressure contours using reflecting boundary conditions, $M_{exit} = 1.1$.

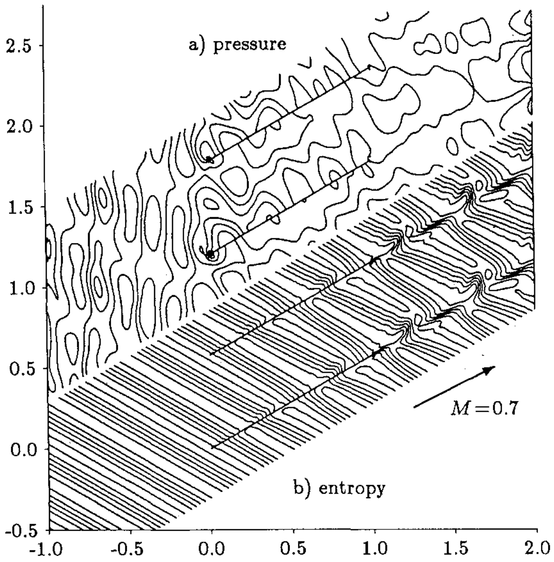

Fig. 4 Flat plate pressure and entropy contours.

boundaries. The results are almost identical. Figure 2 shows the corresponding results for a supersonic outflow condition that has two weak, oblique shocks extending downstream from the trailing edge. The agreement in this case in not quite as good, due to second-order nonlinear effects that are not considered by the linear theory. However, under the standard boundary conditions that impose uniform exit pressure, the outgoing shocks produce reflected expansion waves that greatly contaminate the solution on the blade. This behavior is shown in Fig. 3. Thus, the nonreflecting boundary conditions give a major improvement in accuracy.

B. Unsteady, Nonreflecting Boundary Conditions

The unsteady test case is a relatively simple linear flow consisting of the addition of a low-amplitude sinusoidal wake to a steady uniform flow past an unloaded flat plate cascade. This case was chosen because the results can be compared to those obtained using LINSUB, a program developed by Whitehead,[12] based on the linear singularity theory of Smith.[13] The steady flow has a Mach number of 0.7 and a flow angle of 30 deg, parallel to the flat plates that have a pitch/chord ratio of 0.577. The unsteady wakes have a pitch that is a factor 0.9 smaller, and an angle of -30 deg that corresponds to the outflow angle relative to the upstream blade row. Figure 4 shows contour plots of the entropy and pressure at one instant in time.

To obtain a quantitative comparison, the unsteady pressures from UNSFLO were Fourier transformed, and then nondimensionalized in exactly the same manner as in LINSUB. Figure 5 shows the real and imaginary components of the complex amplitude of the first Fourier mode of the pres-

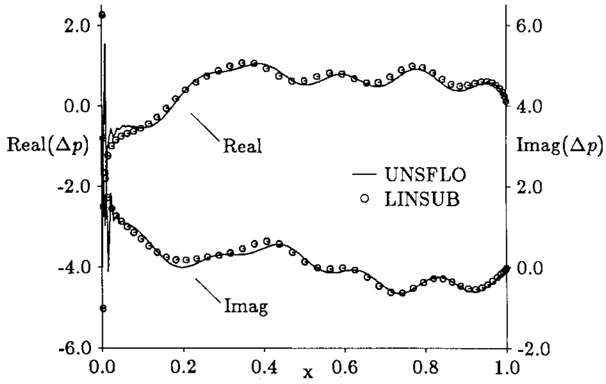

Fig. 5 Complex amplitude of flat plate pressure jump.

sure jump across the blade. The agreement between the UNSFLO computation and the LINSUB theory is good, except at the leading edge where the $x^{-1/2}$ singularity causes some minor oscillations. The integrated lift and moment also agree to within 5%. This test case shows that the computational method is capable of correctly predicting the unsteady forces, due to a wake/vortex interaction. However, the level of agreement is no poorer if the standard quasi-one-dimensional boundary conditions are used. It appears that in this case it is the truncation error of the numerical scheme on the interior of the domain that is responsible for the dominant error term. Thus, unlike the steady-state case, we are unable at present to demonstrate any large improvements in accuracy due to the improved, unsteady, nonreflecting boundary conditions.

V. Conclusions

A unified linear theory for the construction of nonreflecting boundary conditions has been developed and applied to the Euler equations.

Analytically, the steady-state boundary conditions are exact, within the assumptions of the linear theory, which means that in practice the only errors are second-order nonlinear effects that are quadratic in the amplitude of the nonuniformity at the inflow or outflow boundary. Numerical results show that they are extremely effective in a turbomachinery application.

The unsteady boundary conditions are based on a second-order approximation of the ideal nonreflecting boundary conditions, whereas the standard "quasi-one-dimensional" boundary conditions correspond to the first-order approximation. This means that if the wavecrests of outgoing waves are at an angle θ to the boundary, then the amplitude of the artificially reflected wave is $O(\theta^2)$ for the new boundary conditions, as opposed to $O(\theta)$ for the standard boundary conditions. However, numerical results are unable to demonstrate this improvement, due to the dominance of the truncation error of the numerical algorithm.

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